an asymptotic deviation of at most about 22% [4]. This is fairly good and sufficient for most applications. (To further improve on this – at the cost of a considerable increase in running time – there also is an asymptotic "fully polynomial time approximation scheme" based on linear programming methods [7, p. 1574].)

We can apply the first-fit algorithm to ensemble packing using the two strategies sketched out above, as follows:

Simultaneous First-Fit (SFF) Algorithm:

$$\mathcal{B}_v = \mathrm{FF}(R_v, M) \;\forall\, v \in V, \tag{5}$$

Global First-Fit (GFF) Algorithm:

$$\mathcal{B}_v = \{B \in \mathrm{FF}(R_V, M) : B \cap R_v \neq \emptyset\} \;\forall\, v \in V. \tag{6}$$

5 Lower Bounds

Whenever heuristics are employed to solve difficult problems in combinatorial optimization, quality control becomes an important issue. For a minimization problem, we need a good *lower* bound on the optimum value of the target function which can be computed in a reasonable amount of time. With such a bound at hand, we can estimate the maximum deviation of the heuristic solution from the optimum. For the ensemble planning problem, a useful lower bound is provided by an appropriate generalization of the clique number, which we discuss in this section.

As in the preceding section, we let $R_W = \bigcup_{w \in W} R_w \;\forall\, W \subseteq V$. Similarly, the set of all ensembles in a given area set $W \subseteq V$ is denoted $\mathcal{B}_W$, i.e., $\mathcal{B}_W = \bigcup_{w \in W} \mathcal{B}_w$. Our bound is based on the following simple observation:

Lemma 1 (Packing Lemma) *If $\mathcal{B}$ is an admissible ensemble assignment w.r.t. R and M, then $|\mathcal{B}_W| \geq p_M(R_W) \;\forall\, W \subseteq V$.*

Lemma 1 follows immediately from the fact that if $\mathcal{B}$ is admissible, then $R_W \subseteq \bigcup_{B \in \mathcal{B}_W} B \;\forall\, W \subseteq V$ and hence $\mathcal{B}_W$ is an "M-cover" of a superset of R_W (which is just like an M-packing, except that the members of $\mathcal{B}_W$ are not necessarily disjoint). Now consider the special case that W is a clique of G. In this case the subgraph of $G^{\mathcal{B}}$ induced by $\mathcal{B} \cap (W \times \mathcal{B}_W)$ always contains a clique of size $|\mathcal{B}_W|$. (For each $B \in \mathcal{B}_W$ choose some $w_B \in W$ s.t. $B \in \mathcal{B}_{w_B}$. Then $\{(w_B, B) : B \in \mathcal{B}_W\}$ is a clique of the requested size.) Hence

$$p_M(R_W) \leq |\mathcal{B}_W| \leq \omega(G^{\mathcal{B}}) \leq \chi(G^{\mathcal{B}}). \tag{7}$$

Now we define

$$\pi_M^R(G) = \max\{p_M(R_W) : W \text{ clique of } G\}. \tag{8}$$

By taking the maximum over all cliques W of G on the left-hand side of Equation (7), and the minimum over all admissible ensemble assignments $\mathcal{B}$ on the right-hand side, we obtain

$$\pi_M^R(G) \leq \chi_M^R(G). \tag{9}$$